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NYO-7967

BUCKLING OF
SHALLOW SPHERICAL SHELLS
UNDER EXTERNAL PRESSURE

by

Edward L. Reiss

April 1, 1957

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ABSTRACT

A formula for the initial buckling loads for clamped, shallow spherical shells under uniform external pressure is obtained by combining the solutions of two linearized versions of the original nonlinear problem. One of these versions is a linear eigenvalue problem, while the other is the bending problem for a shallow cap in the linear theory of elasticity. The formula, which is obtained in a simple manner, yields buckling loads that are in better agreement with experiments than previous approximate solutions to the nonlinear problem.

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SYMBOLS

E	= Young's Modulus
$2h$	= Shell thickness
k	= $2/3 (1 - \nu^2)$
p	= External pressure, positive inwards
p_{cr}	= Initial buckling pressure
P	= Loading parameter, see equation [4]
P_{cr}	= Value of loading parameter corresponding to p_{cr}
R	= Radius of middle surface of shell
v	= Vertical displacement
w	= Radial component of displacement
γ	= Reduced slope of middle surface = $\alpha - \theta$
$\bar{\alpha}$	= Slope of tangent plane to deformed surface with the horizontal plane
α	= $\bar{\alpha}/\Lambda$
Γ	= $\gamma + P \theta$
γ	= Stress function, see equation [2]
$\bar{\theta}$	= Polar angle
θ	= $\bar{\theta}/\Lambda$
Λ	= Semi-angle of shell opening
ν	= Poisson's ratio
ρ	= Geometric parameter, see equation [5]
σ	= γ/θ
σ^o	= constant
$\sigma_{\bar{\theta}}^o, \sigma_{\bar{\varphi}}^o$	= Longitudinal and circumferential stress on middle surface
$\bar{\varphi}$	= Azimuthal angle
ω^2	= P_p

Buckling of Shallow Spherical Shells
under External Pressure

1. Introduction

Consider a shallow spherical shell, clamped along the edge and under the action of uniform external pressure. For values of pressure increasing from zero, the shell will at first deform in a rather continuous manner. This process persists until the external pressure, p , reaches a certain critical value, p_{cr} , called the initial buckling load. At this value of pressure the shell no longer deforms in a continuous manner but jumps or snaps into another non-adjacent equilibrium configuration. The pressure to which the shell jumps is called the final buckling load. The purpose of this paper is to determine the values of p_{cr} for various shell configurations.

A series of experiments have been conducted by Kaplan and Fung (1) on clamped, shallow, spherical shells under uniform external pressure. Not only were the initial and final buckling loads determined, but measurements were made of the vertical deflections at various points of the shell for several different values of the pressure. As previously noted (2,3), there is an interesting interpretation of these experimental results. In Figure 2, three graphs of the vertical deflection v versus the

polar angle $\bar{\theta}$ are shown. For low values of ρ (a parameter that characterizes the geometry of the shell, see equation [5] and the text below this equation for the definition of ρ), the shell deformed as in Figure 2a, with a maximum deflection at the center $\bar{\theta} = 0$. This will be called mode of deformation I. For ρ in the range $20 < \rho < 55$, the shell deformed in mode of deformation II, as shown in Figure 2b. This mode has the maximum deflection point between the center and the edge of the shell. For values of ρ greater than 55, mode III was observed, see Figure 2c. In this mode, the maximum deflection is again at the center. Two important facts about this behavior are: there exist certain transitional values of ρ , ρ equals approximately 20 and 55, for which the mode of deformation changes; and, associated with a change in mode, there is a change in sign of the curvature at $\bar{\theta} = 0$ for the graphs of deflection versus $\bar{\theta}$. That is, in Figure 2, the curvatures at $\bar{\theta} = 0$ for modes I and III are positive, while for mode II the curvature is negative. Therefore there should be values of ρ for which the curvature at the center vanishes. These ρ 's are defined as the transitional values, ρ_i^* ($i = 1, 2, 3, \dots$).

When the experimental initial buckling loads are plotted against ρ rather than the geometric parameter λ of Kaplan and Fung (1) ($\rho = (2)^{-1/2} \lambda^2$), it is

found (2) that there is an oscillating or peaking behavior of these buckling loads as a function of ρ , see Figure 3. In Figure 3 P_{cr} is the critical value of the loading parameter, related to the physical pressure, p , by equation [4]. The circles are the experimental values, while the solid curve indicates the trend in the data. It is important to notice that the peaks in the initial buckling loads occur for those values of ρ at which there is change in mode of deformation, i.e., for the transitional values ρ_1^* .

Several approximate solutions have been given for the non-linear boundary value problem governing the deformations of a clamped, shallow, spherical shell. The results for a perturbation type of solution, with the non-dimensionalized radial displacement at some point as perturbation parameter (4), and a power series solution (3), are also shown in Figure 3. The initial buckling loads predicted by the perturbation solution and illustrated by the dashed line are in agreement with the experiments only in the neighborhood of $\rho = 20$. For increasing values of ρ , the agreement becomes poorer. Furthermore, the perturbation solution fails to reveal the peaking behavior of the buckling loads as a function of ρ .

The results from the power series solutions are indicated by the short vertical lines in Figure 3. The

value of the buckling load for a prescribed ρ lies somewhere in the open interval spanned by the vertical line. It can be observed that the power series solution yields buckling loads which are in good quantitative agreement with experiments. In addition the peaking trend in the buckling loads is also clearly evident from this solution. The disadvantage of the power series approach is that it yields discrete results rather than a continuous variation with ρ . That is, buckling loads can be computed for only one value of ρ at a time. Furthermore, for larger ρ and P , the number of terms in the series and the number of iterations required for convergence of the numerical procedure become quite large. The calculations thus become rather cumbersome.

In this paper a simplified approach to predicting initial buckling loads is presented. This approach is based upon the solutions to two linearized versions of the original non-linear problem. One of the linearized versions is an eigenvalue problem while the other corresponds to the linear elasticity problem for the bending of a shallow spherical shell. Although the method is rather crude, it yields, in a simple manner, results which are in excellent agreement with the experiments. Finally, a semi-empirical formula is given for determining initial buckling loads.

2. The Nonlinear Problem

The definitions of geometrical quantities associated with the spherical cap are shown in Figure 1. The polar angle $\bar{\theta}$, and the angle $\bar{\alpha}$ that the tangent plane to the deformed surface makes with the horizontal ($\bar{\alpha} = \frac{1}{R} \frac{dw}{d\bar{\theta}} + \bar{\theta}$), may be normalized as follows:

$$[1] \quad \bar{\theta} = \mathcal{L} \theta \quad ; \quad \bar{\alpha} = \mathcal{L} \alpha$$

where $\mathcal{L}$ is the semi-angle of opening of the shell. If a stress function γ is introduced such that

$$[2] \quad \sigma_{\bar{\theta}}^0 = \frac{Ek^{1/2}}{2} \frac{h}{R} \frac{\gamma}{\theta} \quad ; \quad \sigma_{\bar{\varphi}}^0 = \frac{Ek^{1/2}}{2} \frac{h}{R} \frac{d\gamma}{d\theta}$$

where $\sigma_{\bar{\theta}}^0$ and $\sigma_{\bar{\varphi}}^0$ are the longitudinal and circumferential membrane stresses, then the nonlinear differential equations for the rotationally symmetric deformations of a spherical cap are as given in (3)

$$[3a,b] \quad L\alpha = \rho(\alpha\gamma + P\theta^2) \quad ; \quad L\gamma = \rho(\theta^2 - \alpha^2)$$

L is the linear differential operator,

$$L \equiv \theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta$$

and P and ρ , the loading and geometrical parameters, are defined by the following relations,

$$[4] \quad p = 2Ek^{1/2} \left(\frac{h}{R}\right)^2 P$$

$$[5] \quad \rho = (k)^{-1/2} \mathcal{L}^2 \frac{R}{h}$$

where

$$k = \frac{2}{3(1-\nu^2)}$$

and p is the uniform external pressure. The geometrical parameter ρ is also proportional to the ratio of the shell rise and the thickness.

Associated with the differential equations [3a,b] are the following boundary conditions: At the center, $\theta = 0$, there is rotational symmetry; while at the edge, $\theta = 1$, the shell is clamped and the displacement is zero. Mathematically this may be expressed (see reference (3)) as

$$[6a] \quad \alpha(0) = 0 \quad ; \quad \gamma(0) = 0$$

$$[6b] \quad \alpha(1) = 1 \quad ; \quad \left[\frac{d\gamma}{d\theta} - \nu\gamma \right]_{\theta=1} = 0$$

Thus, the complete formulation of the nonlinear boundary value problem consists of the differential equations [3a,b] and the boundary conditions [6a,b]. In reference (3) an approximate solution based upon a power series expansion of α and γ in terms of θ , has been presented for a large range of P and ρ .

3. The Linear Problems

Rather than attempt a direct solution to the nonlinear boundary value problem, the initial buckling loads P_{cr} are determined from the solutions to two linearized problems associated with equations [3a,b] and [6a,b]. The first of these problems, which is a linear eigenvalue problem, will be called Problem A; the second one, which is the linear elasticity problem for the bending of a shallow spherical shell, will be called Problem B.

Problem A:

To obtain the differential equations for Problem A, it is convenient to make the following change of dependent variables:

$$[7] \quad \alpha = y + \theta \quad \gamma = \sigma \theta$$

so that y is proportional to the slope. Then the equilibrium equation, [3a], becomes

$$[8] \quad Ly = \theta \rho (y\sigma + \sigma\theta + P\theta) \quad .$$

It will now be assumed that σ may be written as

$$[9] \quad \sigma = \sigma^0 + \delta\sigma(\theta)$$

where σ^0 is a constant, and $\delta\sigma$ is small compared to σ^0 . This is equivalent to assuming, by equations [2] and [7], that the stress is constant correct to higher order terms. From the results of the numerical solution of (3), this is not an unreasonable assumption.

By substituting [9] into [8] and neglecting higher order terms, there results

$$[10] \quad \frac{d^2 y}{d\theta^2} + \frac{1}{\theta} \frac{dy}{d\theta} + (-\rho \sigma^0 - \frac{1}{\theta^2}) y = \theta (\sigma^0 + P)$$

The second of the boundary conditions of [6b] is replaced by the condition that the total vertical force along the edge $\theta = 1$ is equal to the vertical force produced along the edge by the external pressure. This condition yields the result that at $\theta = 1$,

$$[11] \quad \sigma^0 = -P$$

Setting

$$[12] \quad \omega^2 = P\rho$$

and by substitution of [11] and [12], equation [10] becomes

$$[13] \quad \frac{d^2 y}{d\theta^2} + \frac{1}{\theta} \frac{dy}{d\theta} + (\omega^2 - \frac{1}{\theta^2}) y = 0$$

The linear boundary value problem formed by equation [13] and the boundary conditions,

$$[14] \quad y(0) = y(1) = 0$$

obtained from [6a,b] with the aid of [7], provides a rough approximation to the original nonlinear problem [3a,b] and [6a,b].

The assumptions that have been made in deriving this approximating problem are based on a consideration of an analogous behavior between the spherical shell

under external pressure and the beam-column. The analogy is not complete in the sense that in the beam-column, the lateral and end loads may be varied independantly. While in the shell, the "edge load" at $\theta = 1$ is dependent on the external pressure P . Essentially, the assumption [9] and the condition [11] replace the clamped spherical shell under external pressure, in a first approximation, by a shell that is clamped along the edge $\theta = 1$, and loaded by a uniform force proportional to σ^0 . This shell deforms by contracting into a smaller spherical segment ($\sigma = \text{const.}$, $y = 0$) until some critical value, P_{cr} , of σ^0 is reached. At this value of σ^0 the shell may buckle into non-spherical states.

This eigenvalue problem, defined by equations [13] and [14], has non-trivial solutions

$$[15] \quad y = C J_1(\omega \theta) ,$$

for a discrete set of values, ω_n ($n = 1, 2, \dots$), of ω satisfying the equation

$$[16] \quad J_1(\omega_n) = 0$$

where $J_1(\omega \theta)$ are the Bessel functions of the first kind of order one. The trivial solution satisfies the compatibility equation [3b] under assumption [9].

Therefore, from the definition [12], the initial buckling loads may be expressed as a function of ρ through,

$$[17] \quad P_{cr} = \frac{\omega_n^2}{\rho}, \quad n = 1, 2, 3, \dots$$

In the P_{cr} , ρ plane equation [17] yields a family of equilateral hyperbolas. Each member of the family represents a different mode of deformation. The problem now is to determine those values of ρ for which the solution moves from one member of the family to another, i.e., for which values of ρ there is a transition from the ω_{n-1} hyperbola to the ω_n hyperbola. Since these ρ 's represent the point of change from one mode to another, they are the transitional values ρ_1^* . They will be determined from the solutions of Problem B.

Problem B:

This is the linear elasticity problem for the bending of a shallow spherical shell under uniform external pressure. By considering infinitesimal displacements from equilibrium, neglecting higher order quantities in equation [3a,b], and then substituting

$$u = y + \theta \quad \text{and} \quad \gamma = \Gamma - P\theta$$

into the resulting differential equations and the boundary conditions [6a,b], the following boundary value problem of the linear bending theory is obtained:

$$[18a,b] \quad \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta y - \rho \Gamma = 0; \quad \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta \Gamma + 2 \rho y = 0$$

$$[19a] \quad y(0) = 0; \quad \Gamma(0) = 0$$

$$[19b] \quad y(1) = 0; \quad \left[\frac{d\Gamma}{d\theta} - \nu \Gamma \right]_{\theta=1} = (1-\nu) P$$

The solution to this problem $y = y_B(\theta; P, \rho)$ may be obtained in terms of the Kelvin functions $\text{ber } \theta$ and $\text{bei } \theta$ ($\text{ber } \xi = \text{Re} J_0(i^{3/2}\xi)$, $\text{bei } \xi = \text{Im} J_0(i^{3/2}\xi)$). It depends parametrically on P and ρ . The mode of deformation is independent of P , and varies only with the geometrical parameter ρ . The ρ_i^* have previously been defined as those values of ρ for which the curvature of the deformed state vanishes at $\theta = 0$. Since y is proportional to the slope, the ρ_i^* are the roots of

$$[20] \quad \left. \frac{dy_B}{d\theta} \right|_{\theta=0} = 0 .$$

For the boundary conditions [19a,b] the condition [20] reduces to

$$[21] \quad \text{ber}' Z_1 = 0$$

where

$$[22] \quad Z_1 = .2^{1/4} (\rho_i^*)^{1/2} .$$

The first three roots of [21], (5), yield the transitional values

$$[23] \quad \rho_1^* = 25.8 \quad , \quad \rho_2^* = 78.2 \quad , \quad \rho_3^* = 158.5 \quad .$$

The two linear problems A and B are independent in the sense that they do not supply contradictory information in the same area. Problem A was formulated so as to obtain approximate values of the buckling loads

considered as a bifurcation point in the solution. It therefore could not be expected to yield accurate information concerning deformations. On the other hand, Problem B should yield reasonable results concerning modes of deformation and cannot be expected to supply results with respect to the bifurcation phenomenon. Combining the solutions to Problems A and B, equations [17] and [23], the initial buckling loads for a clamped spherical cap may be evaluated from the following formula:

$$[24] \quad P_{cr} = \begin{cases} \omega_1^2/\rho & \text{for } 0 \leq \rho \leq 25.8 \\ \omega_2^2/\rho & \text{for } 25.8 < \rho \leq 78.2 \\ \omega_3^2/\rho & \text{for } 78.2 < \rho \leq 158.5 \end{cases}$$

where (5)

$$[24a] \quad \omega_1 = 3.832 \quad , \quad \omega_2 = 7.016 \quad , \quad \omega_3 = 10.174$$

4. Comparison with Previous Results and Experiments

The values of the initial buckling loads, equation [24], obtained from the approximating linear theories are indicated by the solid curve in Figure 4. For comparison, the results of the perturbation and power series solutions as well as the experimental results of Kaplan and Fung are also shown in Figure 4. It can be observed that the simplified method given in this paper yields results which are, especially in comparison with previous solutions, in excellent agreement with the experiments. The major point of disagreement with the experiments are the numerical values of ρ_1^* . The values computed by this theory are greater than those observed experimentally. Two possible explanations of this are:

1) The ρ_1^* , equation [23], are computed for a shell that is clamped on the boundary, see [19a,b]. In a physical experiment it is impossible to provide exact clamping on the edge; the actual condition on the edge is somewhere between clamped and simply supported.

2) The differential equations [18a,b] which were employed in computing the ρ_1^* are based on the assumption of a shallow shell. The resulting ρ_1^* [23] may be in error for the larger ρ , since these values of ρ correspond to shells that are non-shallow.

To illustrate point 1), consider Problem B for a simply supported edge. With the boundary conditions

[19a,b] replaced by those of simple support, it can be shown that the equation corresponding to [21] whose roots yield the ρ_1^* for this case is:

$$[25] \quad \sqrt{2} Z_1 \operatorname{bei} Z_1 + (1-\nu)(\operatorname{ber}_1 Z_1 + \operatorname{bei}_1 Z_1) = 0 ,$$

where Z_1 is defined in equation [22]. The first two roots of this equation are

$$[26] \quad \rho_1^* = 17 \quad , \quad \rho_2^* = 62.3 \quad .$$

Thus by considering a non-clamped edge, the values of ρ_1^* are decreased. It may be noticed from Figure 4 that the experimental ρ_1^* is bracketed between the clamped and simply supported ρ_1^* . This justifies the conjecture concerning the effect of variation of boundary conditions between theory and experiment. The further disagreement between theory and experiment in ρ_2^* is probably caused by the non-shallowness of the shell for this value of ρ .

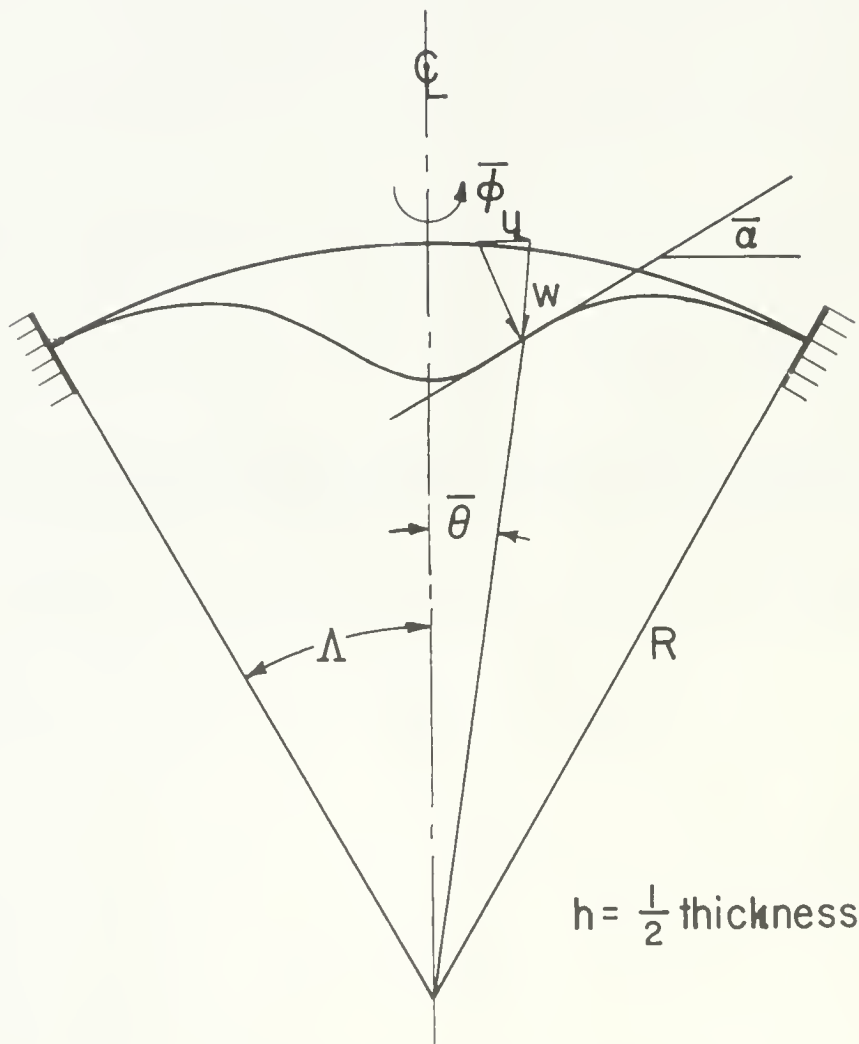
On the basis of this evidence, the following semi-empirical formula is offered in place of [24] for computing the initial buckling load for clamped, shallow spherical shells:

$$[27] \quad P_{cr} = \begin{cases} \omega_1^2/\rho & \text{for } 0 \leq \rho \leq 20 \\ \omega_2^2/\rho & \text{for } 20 < \rho \leq 56 \\ \omega_3^2/\rho & \text{for } 56 < \rho < N \end{cases}$$

where, ω_n are given by equation [24a], and N is to be determined from experiments.

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SHELL GEOMETRY

FIG. 1

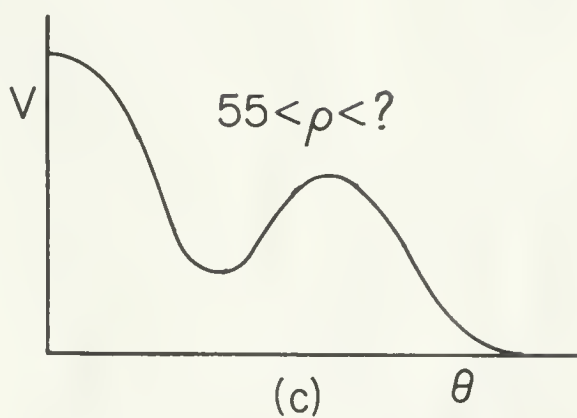
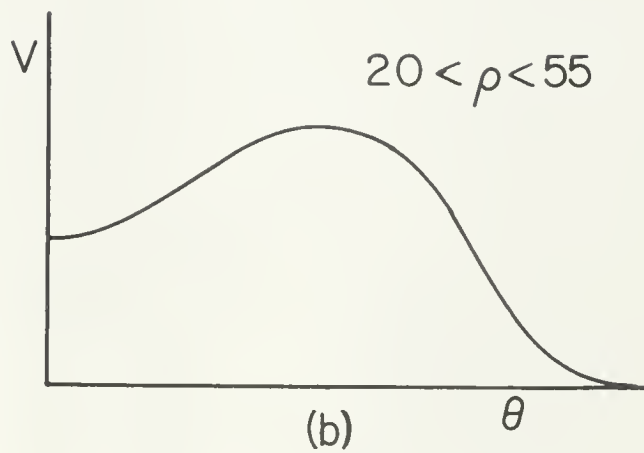
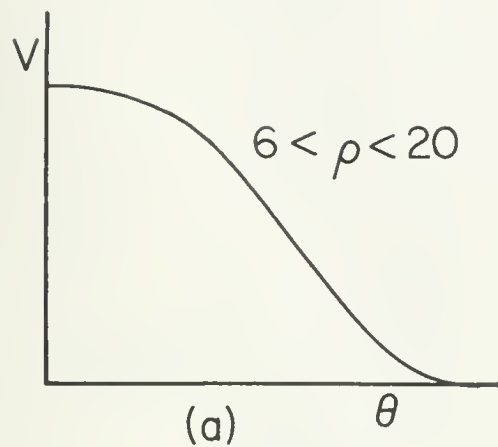


FIG. 2

MODES OF DEFORMATION

COMPARISON OF PREVIOUS THEORETICAL AND EXPERIMENTAL INITIAL BUCKLING LOADS

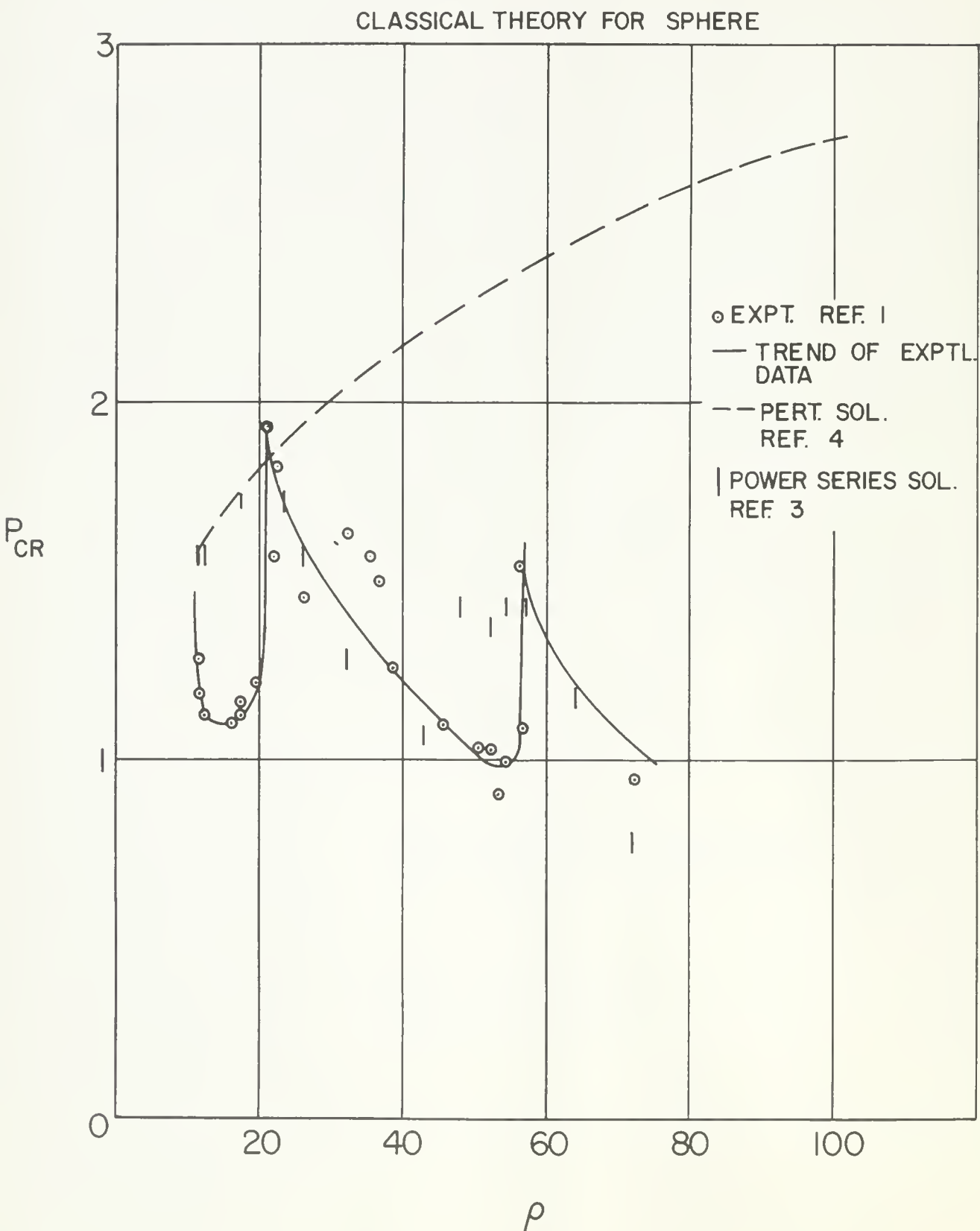


FIG. 3

COMPARISON OF INITIAL BUCKLING LOADS

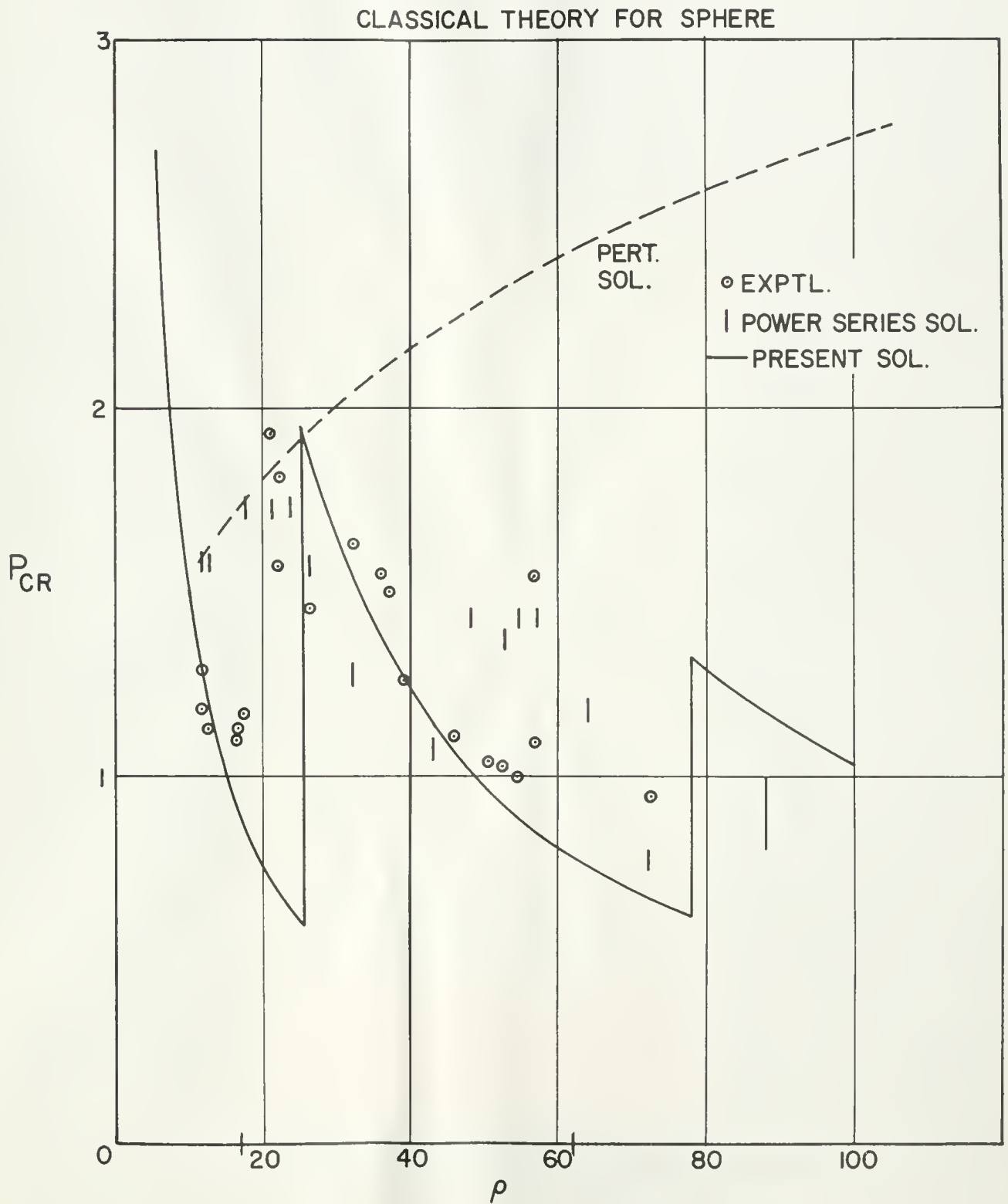


FIG. 4

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